induced by this topology is denoted by $\varkappa_K(X)$. The spaces $\varkappa_{Cv}(X)$ and $\varkappa_{Kv}(X)$ are defined similarly.

1.1.2. The Lower Semifinite Topology. The Space $\lambda_C(X)$. An open prebasis of this topology is formed by the set $C(X) \setminus C(X \setminus U)$, where U runs through the collection of all open subsets of the space X. This is the weakest topology in which the sets C(F), where F is closed in X, are closed. The set C(X) equipped with this topology is denoted by $\lambda_C(X)$. The spaces $\lambda_K(X)$, $\lambda_{Cv}(X)$, $\lambda_{Kv}(X)$ are defined similarly.

1.1.3. The Exponential Topology. The Space $\mathrm{Exp}_C(X)$. An open prebasis of this topology is formed by the family of all subsets C(U) and all subsets $C(X) \setminus C(X \setminus U)$, where U is open in X. The set C(X) equipped with this topology is denoted by $\mathrm{Exp}_C(X)$. The spaces $\mathrm{Exp}_K(X)$, $\mathrm{Exp}_{Cv}(X)$, $\mathrm{Exp}_{Kv}(X)$ are defined similarly.

1.1.4. The Hausdorff Metric. The Space $M_{Cb}(X)$. Let (X, ρ) be a metric space; Cb(X) denotes the collection of all nonempty, closed, bounded subsets of X. For $A_1, A_2 \in Cb(X)$ let

$$h(A_1, A_2) = \inf\{\varepsilon \mid \varepsilon \geqslant 0,\ A_1 \subset U_\varepsilon(A_2),\ A_2 \subset U_\varepsilon(A_1)\},$$

where $U_\varepsilon(A_i)$ is an ε-neighborhood of the set A_i; $i = 1, 2$.

The function h satisfies all the axioms of a metric on Cb(X) and is called the Hausdorff metric; the metric space (Cb(X), h) is denoted by $M_{Cb}(X)$.

In the case where X is a compact metric space the topology generated by the Hausdorff metric on Cb(X) is equivalent to the exponential topology on this set.

2. Continuity of Multivalued Mappings. Some Operations on Multivalued Mappings

A multivalued mapping F of a set X into a set Y is a correspondence assigning to each point $x \in X$ a nonempty subset $F(x) \subset Y$ called the image of the point x, i.e., this is a single-valued mapping $F: X \to P(Y)$. Henceforth any mapping $F: X \to P(Y)$ is called an m-mapping.

Let X, Y be topological spaces.

1.2.1. Definition. An m-mapping $F: X \to P(Y)$ is called upper semicontinuous at a point $x \in X$ if for any open neighborhood V of the set F(x) there is an open neighborhood U of the point x such that $F(U) \subset V$.

An m-mapping $F: X \to P(Y)$ is upper semicontinuous if it is upper semicontinuous at each point $x \in X$.

We introduce the following notation:

$$F_+^{-1}(D) = \{x \in X \mid F(x) \subset D\},$$

$$F_-^{-1}(D) = \{x \in X \mid F(x) \cap D \neq \varnothing\}.$$

1.2.2. THEOREM. The following conditions are equivalent:

(a) an m-mapping F is upper semicontinuous;

(b) for any open $V \subset Y$ the set $F_+^{-1}(V)$ is open in X;

(c) for any closed $W \subset Y$ the set $F_-^{-1}(W)$ is closed in X;

(d) for any $D \subset Y$ we have $F_-^{-1}(\bar{D}) \supset \overline{F_-^{-1}(D)}$.

1.2.3. THEOREM. An m-mapping $F: X \to C(Y)$ is upper semicontinuous if and only if it is continuous as a mapping into $\varkappa_C(Y)$.

1.2.4. Definition. An m-mapping $F: X \to P(Y)$ is called lower semicontinuous at a point $x \in X$ if for any $V \subset Y$ such that $F(x) \cap V \neq \varnothing$, there is an open neighborhood U of the point x such that $F(x') \cap V \neq \varnothing$ for any $x' \in U$. An m-mapping $F: X \to P(Y)$ is called lower semicontinuous if it is lower semicontinuous at each point $x \in X$.

1.2.5. THEOREM. The following conditions are equivalent:

(a) an m-mapping F is lower semicontinuous;

(b) for any open $V \subset Y$ the set $F_-^{-1}(V)$ is open in X;

(c) for any closed $W \subset Y$ the set $F_+^{-1}(W)$ is closed in X;

(d) for any $D \subset Y$ we have $F_+^{-1}(\bar{D}) \subset \overline{F_+^{-1}(D)}$;

(e) for any $A \subset X$ we have $F(\bar{A}) \subset \overline{F(A)}$.

1.2.6. THEOREM. An m-mapping $F:X \to C(Y)$ is lower semicontinuous if and only if it is continuous as a mapping into $\lambda_C(Y)$.

1.2.7. Definition. If an m-mapping $F:X \to P(Y)$ is upper and lower semicontinuous, then it is called continuous.

1.2.8. THEOREM. An m-mapping $F:X \to C(Y)$ is continuous if and only if it is continuous as a mapping into $Exp_C(Y)$.

1.2.9. Definition. An m-mapping $F:X \to C(Y)$ is called closed if its graph

$$\Gamma_F = \{(x, y) \in X \times Y \mid y \in F(x)\}$$

is a closed set in $X \times Y$.

1.2.10. THEOREM. The following conditions are equivalent:

a) an m-mapping F is closed;

b) for any pair $x \in X$, $y \in Y$ such that $y \bar{\in} F(x)$, there exist neighborhoods $U(x)$ of the point x and $V(y)$ of the point y such that $F(U(x)) \cap V(y) = \varnothing$;

c) for any filters $\{x_\alpha\} \subset X$, $\{y_\alpha\} \subset Y$ such that $x_\alpha \to x$, $y_\alpha \in F(x_\alpha)$, $y_\alpha \to y$, we have $y \in F(x)$. (In the case of metric spaces it suffices to consider ordinary sequences.)

There is a close relation between closed and upper semicontinuous m-mappings.

1.2.11. THEOREM. If an m-mapping $F:X \to C(Y)$ is upper semicontinuous and the space Y is regular, then F is closed. If an m-mapping F has compact range, $F:X \to K(Y)$, then in this assertion the condition of regularity of Y can be relaxed: it suffices to require that it be Hausdorff.

1.2.12. THEOREM. If $F:X \to K(Y)$ is a closed locally compact m-mapping, then it is upper semicontinuous.

We note the following properties of closed and upper semicontinuous m-mappings.

1.2.13. THEOREM. If an m-mapping $F:X \to C(Y)$ is closed and $A \in K(X)$, then $F(A) \in C(Y)$.

In the case where Y is a metric space it is possible to give the following simple criteria for the various types of continuity of m-mappings.

1.2.15. THEOREM. For upper (lower) semicontinuity of an m-mapping $F:X \to K(Y)$ at a point $x \in X$ it is necessary and sufficient that for any $\varepsilon > 0$ there exist a neighborhood $U(x)$ of the point x such that $F(x') \subset U_\varepsilon(F(x))$ [respectively, $F(x) \subset U_\varepsilon(F(x'))$] for all $x' \in U(x)$.

1.2.16. Definition. An m-mapping $F:X \to Cb(Y)$ is called continuous in the Hausdorff metric if F is continuous as a mapping into the metric space $(Cb(Y), h)$.

1.2.17. THEOREM. An m-mapping $F:X \to K(Y)$ is continuous if and only if it is continuous in the Hausdorff metric.

Some operations on multivalued mappings and the properties of continuity connected with them are described below.

Let X, Y be topological spaces; let $\{F_j\}_{j \in J}$, $F_j:X \to P(Y)$ be some family of m-mappings.

1.2.18. THEOREM. a) Suppose the m-mappings F_j are upper semicontinuous. If the set of indices J is finite, then the union of m-mappings $\bigcup_{j \in J} F_j : X \to P(Y)$,

$$(\bigcup_{j \in J} F_j)(x) = \bigcup_{j \in J} F_j(x),$$

is upper semicontinuous.

b) Suppose the m-mappings F_j are lower semicontinuous. Then the union $\bigcup_{j \in J} F_j$ is lower semicontinuous.

c) Suppose the m-mappings $F_j:X \to C(Y)$ are closed. If the set of indices J is finite, then the union $\bigcup_{j \in J} F_j : X \to C(Y)$ is closed.

1.2.19. THEOREM. a) Suppose the m-mappings $F_j:X \to C(Y)$ are upper semicontinuous. If the set of indices J is finite, the space Y is normal, and $\bigcap_{j\in J} F_j(x) \neq \varnothing \ \forall x \in X$, then the intersection of m-mappings $\bigcap_{j\in J} F_j \to C(Y)$,

$$\left(\bigcap_{j\in J} F_j\right)(x) = \bigcap_{j\in J} F_j(x),$$

is upper semicontinuous.

b) If the m-mappings $F_j:X \to C(Y)$ are closed and $\bigcap_{j\in J} F_j(x) \neq \varnothing \ \forall x \in X$, then the intersection $\bigcap_{j\in J} F_j: X \to C(Y)$ is closed.

c) Suppose the m-mapping $F_0:X \to C(Y)$ is closed, the m-mapping $F_1:X \to K(Y)$ is lower semicontinuous, and $F_0(x) \cap F_1(x) \neq \varnothing \ \forall x \in X$. Then the intersection $F_0 \cap F_1: X \to K(Y)$ is upper semicontinuous.

1.2.20. Example. Suppose m-mappings $F_1, F_2: [-\pi, \pi] \to \mathbb{R}^2$ are defined by the following relations:

$$F_1(t) = \{x = (x_1, x_2) \mid x_1^2 + x_2^2 \leqslant 1, \ x_2 \geqslant 0\},$$
$$F_2(t) = \{x = (x_1, x_2) \mid x_1 = \lambda \cos t, \ x_2 = \lambda \sin t, \ -1 \leqslant \lambda \leqslant 1\}.$$

Although both these m-mappings are continuous and $F_1(t) \cap F_2(t) \neq \varnothing$ for any $t \in [-\pi, \pi]$, the m-mapping $F_1 \cap F_2$ is not lower semicontinuous.

This example shows that the intersection of lower semicontinuous mappings need not be lower semicontinuous.

Let X, Y, Z be topological spaces.

1.2.21. THEOREM. a) If the m-mappings $F_0:X \to P(Y)$, $F_1:Y \to P(Z)$ are upper (lower) semicontinuous, then their composition

$$(F_1 \circ F_0)(x) = F_1(F_0(x))$$

is upper (lower) semicontinuous.

b) If the m-mapping $F_0:X \to K(Y)$ is upper semicontinuous and the m-mapping $F_1:Y \to C(Z)$ is closed, then their composition $F_1 \circ F_0:X \to C(Z)$ is closed.

Let X be a topological space, and let Y be a topological vector space.

1.2.22. THEOREM. a) If the m-mappings F_0, $F_1:X \to P(Y)$ are lower semicontinuous, then their sum $F_0 + F_1:X \to P(Y)$, $(F_0+F_1)(x) = F_0(x) + F_1(x)$ is lower semicontinuous.

b) If the m-mappings F_0, $F_1:X \to K(Y)$ are upper semicontinuous, then their sum $F_0 + F_1: X \to K(Y)$ is upper semicontinuous.

1.2.23. THEOREM. a) If the m-mapping $F:X \to P(Y)$ is lower semicontinuous and the function $f: X \to \mathbb{R}$ is continuous, then the product $f \cdot F:X \to P(Y)$

$$(f \cdot F)(x) = f(x) \cdot F(x)$$

is lower semicontinuous.

b) If the m-mapping $F:X \to K(Y)$ is upper semicontinuous and the function $f: X \to \mathbf{R}$ is continuous, then the product $f \cdot F:X \to K(Y)$ is upper semicontinuous.

Let Y be a complete, locally convex space (lcs).

1.2.24. THEOREM. If the m-mapping $F:X \to K(Y)$ is upper (lower) semicontinuous, then the convex closure $\overline{\text{co}}\, F:X \to Kv(Y)$

$$(\overline{\text{co}}\, F)(x) = \overline{\text{co}}\,(F(x))$$

is upper (lower) semicontinuous.

In conclusion, we present an assertion called the maximum theorem or the principle of continuity of optimal solutions which plays an important role in applications of multivalued mappings in game theory and mathematical economics.

1.2.25. THEOREM. Let Y, X be topological spaces, let $\Phi:X \to K(Y)$ be a continuous m-mapping, and let $f:X \times Y \to \mathbf{R}$ be a continuous function. Then the function $\varphi: X \to \mathbf{R}$,

$$\varphi(x)=\max_{\tilde{y}\in\Phi(x)} f(x,\tilde{y})$$

is continuous, and the m-mapping $F:X \to K(Y)$

$$F(x)=\{y \mid y\in\Phi(x),\ f(x,y)=\varphi(x)\}$$

is upper semicontinuous.

3. Continuous Sections and Single-Valued Approximations of m-Mappings

Let X, Y be topological spaces, and let $f:X \to Y$ be an m-mapping.

1.3.1. Definition. A continuous, single-valued mapping $f:X \to Y$ is called a continuous section of an m-mapping F if

$$f(x)\in F(x)$$

for all $x\in X$.

The existence of continuous sections is closely connected with lower semicontinuity of a multivalued mapping. The following assertion characterizes this fact.

1.3.2. THEOREM. Let $F:X \to P(Y)$ be an m-mapping. If for any points $x\in X$ and $y\in F(x)$ there exists a continuous section $f:X \to Y$ of the m-mapping F such that $f(x) = y$, then F is a lower semicontinuous m-mapping.

Michael's theorem is one of the basic results of the theory of continuous sections which has found many applications.

1.3.3. THEOREM. The following properties of a T_1-space X are equivalent:

a) X is paracompact;

b) if Y is a Banach space, then each lower semicontinuous m-mapping $F:X \to Cv(Y)$ has a continuous section.

The proof of Theorem 1.3.3 is based on the following assertion.

1.3.4. LEMMA. Let X be a paracompact space, and let Y be a normal space; let $F:X \to Cv(Y)$ be a lower semicontinuous m-mapping; then for any $\varepsilon > 0$ there exists a continuous single-valued mapping $f_\varepsilon:X \to Y$ such that $f_\varepsilon(x)\in U_\varepsilon(F(x))$ for any $x\in X$.

This mapping f_ε is naturally called an ε-section of the m-mapping F.

There are many examples which show that the conditions of completeness of the space Y, closedness and convexity of the range of the m-mapping F, and the condition of lower semicontinuity of this mapping are essential for the existence of a continuous section. However, it is obvious that there exist m-mappings which are not lower semicontinuous but have a continuous section. We shall consider the problem of the existence of a continuous section in terms of the local structure of m-mappings (see [22]).

Let X be a metric space, Y be a convex compact subset of the Banach space E, and let $F:X \to Kv(Y)$ be some m-mapping. We set $F^\varepsilon(x)=U_\varepsilon(F(x))\cap Y$. For each point $x_0\in X$ we define the set $L(F)(x_0)$ by the rule

$$L(F)(x_0)=\bigcap_{\varepsilon>0}\left(\overline{\bigcup_{\delta>0}\left(\bigcap_{x\in U_\delta(x_0)} F^\varepsilon(x)\right)}\right).$$

1.3.5. THEOREM. In order that an m-mapping $F:X \to Kv(Y)$ have an ε-section for any $\varepsilon > 0$ it is necessary and sufficient that $L(F)(x_0) \neq \emptyset$ for any $x_0\in X$.

We remark that nonemptiness of the set $L(G)(x)$ for any $x\in X$ does not yet guarantee the presence of a continuous section of an m-mapping F.

We consider iterations of L:

$$L^0(F)=F,\ L^n(F)=L(L^{n-1}(F)),\ n\geqslant 1.$$

We continue this process for each transfinite number of first type, while for a transfinite number of second type we set

$$L^\alpha(F)(x)=\bigcap_{\beta<\alpha} L^\beta(F)(x).$$